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A LOCAL VERSION OF A RESULT OF GAMLEN AND GAUDET

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ABSTRACT
Let X^p_D be the span of the Haar function $\{h_j : J \in D\}$ in L^p $(1 < p < \infty)$ endowed with L^p norm. Then for any finite set D, the spaces X^p_D and $l^p_{\#D}$ are K_p-isomorphic where K_p depends on p only.

In [G.G] Gamlen and Gaudet consider subspaces of L^p $(1 < p < \infty)$ which are generated by an infinite subset of the Haar basis.

Using the Banach space decomposition principle of Pelczynski, Gamlen and Gaudet proved that l^p and L^p are the only Banach spaces which can be produced in this way.

The argument used below to prove the corresponding finite dimensional result (Theorem 1) is based on the methods of B. Maurey, cf. [Ma, Section 4].

THEOREM 1. *There exists $K_p > 0$ such that the span X^p of any finite subset of the Haar basis in $L^p[0, 1]$ $(1 < p < \infty)$ is K_p-isomorphic to $l^p_{\dim X^p}$.*

Subsets of the Haar basis are given by a collection $\mathscr{D}$ of dyadic intervals. To handle such collections properly we will introduce generations:

For $I \in \mathscr{D}$ we let

$$G_1(I \mid D) = \{ J \in \mathscr{D} : J \subsetneq I, J \text{ maximal}\}.$$

Having defined $G_1(I \mid \mathscr{D}), \ldots, G_{n-1}(I \mid \mathscr{D})$ we put

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$$G_n(I \mid \mathscr{D}) = \bigcup_{J \in G_{n-1}(I \mid \mathscr{D})} G_1(J \mid \mathscr{D}).$$

For arbitrary collection $\mathscr{F}$ of dyadic intervals we denote $\bigcup_{J \in \mathscr{F}} J$ by $\mathscr{F}^*$. When no confusion is possible we will write $G_1(I)$ instead of $G_1(I \mid \mathscr{D})$, and G_n instead of $G_n([0, 1] \mid \mathscr{D})$. For $1 < p < \infty$ we denote by $X^p_{\mathscr{D}}$ the span of the Haar functions $\{h_J : J \in \mathscr{D}\}$ equipped with L^p norm.

Given $f = \Sigma_{J \in \mathscr{D}}\, a_J h_J$ then we put

$$\| f \|_{X^1_{\mathscr{D}}} = \int \left(\sum a_J^2 h_J^2 \right)^{1/2} dt.$$

$X^1_{\mathscr{D}}$ will denote the span of the Haar functions $\{h_J : J \in \mathscr{D}\}$ equipped with the norm $\| \;\; \|_{X^1_{\mathscr{D}}}$.

We let $(\Omega, \mathscr{F}, P)$ be a probability space and we let $(\mathscr{F}_n)_{n \in N}$ be an increasing sequence of finite fields such that

$$\bigvee_{n=1}^{\infty} \mathscr{F}_n = \mathscr{F}.$$

$(\mathbf{E}_n(\cdot))_{n \in N}$ denote the conditional expectations with respect to $(\mathscr{F}_n)$. We introduce the spaces

$$H^p[(\mathscr{F}_n)] := \left\{ f \in L^p(\Omega, \mathscr{F}, \mathbf{P}) : \int S(f)^p < \infty \right\},$$

endowed with the norm

$$\| f \|_{H^p[(\mathscr{F}_n)]} = \left(\int S(f)^p d\mathbf{P} \right)^{1/p}$$

where

$$(Sf)(t) = \left(\sum_{n=1}^{\infty} (\mathbf{E}_n(f) - \mathbf{E}_{n-1}(f))^2 \right)^{1/2} (t).$$

Important to us are the following results:

Theorem (Burkholder). *Let $1 < p < \infty$, then there exists $C_p > 0$ such that for $f \in L^p(\Omega, \mathscr{F}, P)$*

$$\frac{\| f \|_p}{C_p} \leqq \| f \|_{H^p[(\mathscr{F}_n)]} \leqq C_p \| f \|_p.$$

An immediate consequence of Burkholder's theorem is the following:

PROPOSITION 2. *For $1<p<\infty$ there exists $C_p>0$ with the following property:*

For any sequence $(\mathscr{F}_n)$ of increasing finite fields, which is eventually constant, the spaces $H^p[(\mathscr{F}_n)]$ and $l^p_{\dim H^p[(\mathscr{F}_n)]}$ are C_p-isomorphic.

We will also use the following:

THEOREM (E. M. Stein). *Let $1<p<\infty$, then there exists $C_p>0$ such that for any sequence $(f_n)\in L^p(\Omega,\mathbf{P})$*

$$\int\left(\sum|\mathbf{E}_n f_n|^2\right)^{p/2} d\mathbf{P} \leqq C_p \int\left(\sum|f_n|^2\right)^{p/2} d\mathbf{P}.$$

LEMMA 3. *For $1<p<\infty$ there exists $C_p>0$ such that for any finite collection $\mathscr{D}$ of dyadic intervals there exist $N\in\mathbf{N}$ and pairwise disjoint collections $\{\mathscr{D}_n\}_{n\leq N}$ of dyadic intervals such that*

(1) $\mathscr{D}=\bigcup_{n=1}^{N}\mathscr{D}_n$.

(2) *For each $m\in\mathbf{N}$ and $I\in\mathscr{D}_m$*

(a) $|G_1^*(I\mid\mathscr{D}_m)|<|I|$;

(b) *either* $|G_1^*(I\mid\mathscr{D}_m)|>|I|/2$ *or* $|G_1^*(I\mid\mathscr{D}_m)|=0$.

(3) *The Banach–Mazur distance between the space $X^p_{\mathscr{D}}$ and $(\Sigma X^p_{\mathscr{D}_n})_p$ is less than C_p.*

PROOF. We first decompose $\mathscr{D}$ into two collections R and B each of which satisfies condition (a).

Let $\mathscr{L}=\{I\in\mathscr{D}:G_1^*(I\mid\mathscr{D})=|I|\}$; for $I\in\mathscr{L}$ we pick any interval in $G_1(I\mid\mathscr{D})$ which we call $R(I)$. Then we simply set

$$R:=\bigcup\{R(I):I\in\mathscr{L}\},$$

$$B:=\mathscr{D}\setminus R.$$

Now for $I\in B$ the following holds:

$I\setminus G_1^*(I\mid B)$ contains the intersection of a nested family of subsets of R, hence it has positive measure.

The same statement holds with the roles of R and B interchanged, hence for $I\in R$, $I\setminus G_1^*(I\mid R)$ has positive measure.

From now on we assume that $\mathscr{D}$ already satisfed (a).

We now decompose $\mathscr{D}$ inductively.

Step 0

$$\mathscr{I}_1 = \{I \in \mathscr{D} : |G_1^*(I \mid D)| \leqq |I|/2\},$$

$$K_1 = \{I \in \mathscr{D}: \textit{there exists } J \in \mathscr{I}_1 \textit{ and } J \supset I \textit{ and } I \neq J\},$$

$$\mathscr{D}_1 = \mathscr{D} \setminus K_1.$$

If K_1 is empty we stop. If K_1 is nonempty we continue:

Step 1

$$\mathscr{I}_2 = \{I \in K_1 : |G_1^*(I \mid \mathscr{D})| \leqq |I|/2\},$$

$$K_2 = \{I \in K_1: \textit{there exists } J \in \mathscr{I}_2 \textit{ and } J \supset I \textit{ and } J \neq I\},$$

$$\mathscr{D}_2 = K_1 \setminus K_2.$$

Suppose $K_1, \ldots, K_n$, $\mathscr{D}_1, \ldots, \mathscr{D}_n$ are constructed. If K_n is empty we stop. If K_n is nonempty we continue.

Step n

$$\mathscr{I}_{n+1} = \{I \in K_n : |G_1(I \mid \mathscr{D})| \leqq |I|/2\},$$

$$K_{n+1} = \{I \in K_n: \textit{there exists } J \in \mathscr{I}_{n+1} \textit{ and } J \supset I \textit{ and } J \neq I\},$$

$$\mathscr{D}_{n+1} = K_n \setminus K_{n+1}.$$

Obviously $\{\mathscr{D}_n\}$ satisfies (1) and (2); it remains to check (3).

Indeed by construction for each $m \in \mathbf{N}$ and $I \in \mathscr{D}_m$:

$$\left| I \setminus \bigcup_{m+2 \leq n} \mathscr{D}_n^* \right| > \frac{|I|}{2}.$$

Now pick $f \in X^p_{\mathscr{D}}$ and put

$$f_1 = \sum_{n \geq 1} \sum_{J \in \mathscr{D}_{2n-1}} a_J h_J \quad \text{and} \quad f_2 = \sum_{n \geq 1} \sum_{J \in \mathscr{D}_{2n}} a_J h_J.$$

By the unconditionality of the Haar basis in L^p we get

$$\| f \|_p^p > C_p(\| f_1 \|_p^p + \| f_2 \|_p^p).$$

Moreover

$$\| f_2 \|_p^p \geqq C_p \int \Big(\sum_{n \geqq 1} \sum_{J \in \mathscr{D}_{2n}} a_J^2 h_J^2 \Big)^{p/2}$$

$$\geqq C_p \sum_{n \geqq 1} \int \Big(\sum_{J \in \mathscr{D}_{2n}} a_J^2 \chi_{J \setminus \cup_{m>n} \mathscr{D}_{2m}^*} \Big)^{p/2}$$

$$\geqq 2C_p \sum_{n \geqq 1} \Big\| \sum_{J \in \mathscr{D}_{2n}} a_J h_J \Big\|_p^p ,$$

f_1 may be treated the same way, hence there is a constant $C_p > 0$ such that

$$\| f \|_p^p \geqq C_p \Big(\sum_{m \in \mathbf{N}} \Big\| \sum_{J \in \mathscr{D}_m} a_J h_J \Big\|_p^p \Big);$$

on the other hand for $1 \leqq p \leqq 2$, Clarkson's inequality implies

$$\| f \|_p^p \leqq C_p \Big(\sum_{m \in \mathbf{N}} \Big\| \sum_{J \in \mathscr{D}_m} a_J h_J \Big\|_p^p \Big).$$

By duality we get the desired isomorphism for all $1 < p < \infty$.

The rest of the paper is used to show that $X_{\mathscr{D}}^p$ is isomorphic to a certain $H^p[(\mathscr{F}_n)]$ space, provided $\mathscr{D}$ satisfies condition (2) of Lemma 3.

The structure of $\mathscr{D}$ must be reflected by $(\mathscr{F}_n)$ if we want the *norm* of the isomorphism to be independent of $\mathscr{D}$.

Definition 4. Let $\mathscr{D}$ be a finite collection of dyadic intervals, such that for $I \in \mathscr{D}$, $|I| - |G_1^*(I, \mathscr{D})| > 0$. We let $\mathscr{F}_0 = \{[0,1], \varnothing\}$. For $n \geqq 1$, $\mathscr{F}_n$ is defined to be the algebra generated by $\{\mathscr{F}_{n-1} \cup G_n([0,1], \mathscr{D})\}$.

The sequence $(\mathscr{F}_n)$ will be called the filteration induced by $\mathscr{D}$.

For $I \in G_{m-1}$ we put

$$\mathscr{D}(I) := \Big\{ f : I \to \mathbf{R}, \int f = 0, f \textit{ is } \mathscr{F}_n \textit{ measurable} \Big\},$$

$$X(I) := \operatorname{span}\{ |h_J| : J \in G_1(I \mid \mathscr{D})\}.$$

Lemma 5. *There exists $C > 0$ with the following property. If $0 < |I| - |G_1^*(I \mid \mathscr{D})| < |I|/2$, there exists an isomorphism $T_I : X(I) \to \mathscr{D}(I)$ such that, for $1 \leqq p \leqq \infty$ and $f \in X(I)$,*

$$\frac{\| f \|_p}{C} \leqq \| T_I f \|_p \leqq C \cdot \| f \|_p .$$

PROOF (cf. [Ma], Lemma 4.10).

Step 1. Define $m_0 \in \mathbf{N}$ by the relation

$$\frac{|I|}{2.2^{m_0}} < |I| - |G_1^*(I \mid \mathscr{D})| < \frac{|I|}{2^{m_0}} .$$

There exist pairwise disjoint collections $\mathscr{D}_i \subset G_1(I)$, $1 \leqq i \leqq m_0$ such that $\bigcup_{i=1}^{m_0} \mathscr{D}_i = G_1(I)$. Moreover, we can choose them in such a way that for $F_i := \mathscr{D}_i^*$, $1 \leqq i \leqq m_0$ and $F_{m_0+1} := I \setminus G_1^*(I)$ the following holds:

$$|F_1|/|I| = \tfrac{1}{2},$$

$$|F_{i+1}|/|F_i| = \tfrac{1}{2} \qquad \text{for } 1 \leqq i \leqq m_0 - 1,$$

$$\left(\frac{1}{4} + \frac{1}{2}\right) \leqq \frac{|F_{m_0+1}|}{|F_{m_0}|} \leqq \frac{1}{2} .$$

By $\mathscr{C}_I$ we denote the algebra generated in I by $\{F_1, \ldots, F_{m_0+1}\}$. For a $\mathscr{C}_I$ measurable function $f: I \to \mathbf{R}$ we define the shift operator D_I as follows:

$$D_I f_{/F1} := 0,$$

$$D_I f_{/Fi} := f_{/Fi-1} \qquad \text{for } 2 \leqq i \leqq m_0 + 1.$$

We easily observe here that for $1 \leqq p \leqq \infty$ and $f \in X(I)$

$$\| f \|_p / 4 \leqq \| D_I f \|_p \leqq 4 \| f \|_p .$$

Finally we define U_I by

$$U_I f = f - \mathbf{E}(f \mid \mathscr{C}_I) + D_I(\mathbf{E}(f \mid \mathscr{C}_I)).$$

For $1 \leqq p \leqq \infty$, U_I acts as an isomorphism on $L_p(I)$ when restricted to $X(I)$, because for $f \in X(I)$

$$\tfrac{1}{6} \cdot \| f \|_p \leqq \| U_I f \|_p \leqq 6 \cdot \| f \|_p .$$

Indeed, the right-hand inequality is clear. The left-hand inequality needs two observations. Firstly, it follows from the construction that

$$\mathbf{E}(U_I f \mid \mathscr{C}_I) = D_I(\mathbf{E}(f \mid \mathscr{C}_I)).$$

Secondly, we have $f = U_I f + \mathbf{E}(f \mid \mathscr{C}_I) - D_I(\mathbf{E}(f \mid \mathscr{C}_I))$. Hence,

$$\begin{aligned}\| f \|_p &\leqq \| U_I f \|_p + \| \mathbf{E}(f \mid \mathscr{C}_I) \|_p + \| D_I(\mathbf{E}(f \mid \mathscr{C}_I)) \|_p \\ &\leqq \| U_I f \|_p + 5 \| \mathbf{E}(U_I f \mid \mathscr{C}_I) \|_p \\ &\leqq 6 \| U_I f \|_p .\end{aligned}$$

Step 2 (due to G. Schechtman). For $f : I \to \mathbf{R}$ we define V_I by

$$V_I f = f - \left(\int_I f \frac{dt}{|I|} \right) 2\chi_{F_1} .$$

We claim that $\int_{F_1} f = 0$ implies, for $1 \leqq p \leqq \infty$: $\frac{1}{2} \| f \|_p \leqq \| V_I f \|_p \leqq 2 \| f \|_p$.

Indeed, the right-hand side inequality is obvious, and we need only verify the left-hand side inequality. Choose $h \in L^q$ such that $\| h \|_q = 1$ and $\int f \cdot h = \| f \|_{L^p}$. Let

$$\bar{h} := h - \int_{F_1} h \frac{dt}{|F_1|} .$$

We have now

$$\| \bar{h} \|_q \leqq 2 \| h \|_q, \quad \int \bar{h} f = \int h f \quad \text{and} \quad \int (\bar{h} \cdot \chi_{F_1}) = 0.$$

Hence

$$\begin{aligned}\| V_I f \|_p &\geqq \frac{1}{2} \int \bar{h} V_I f \\ &= \frac{1}{2} \int \bar{h} f + \frac{1}{2} \left(\int \bar{h} \chi_{F_1} \right) \cdot \left(\int_I f dt \right) \cdot 2 \\ &= \frac{1}{2} \int \bar{h} f = \frac{1}{2} \| f \|_p .\end{aligned}$$

Step 3. Fix $f \in X(I)$ (i.e. $f = \sum_{J \in G_1(I)} a_J h_J$). Then we define P_I by

$$P_I f = \sum_{J \in G_1(I)} a_J \chi_J .$$

Finally we put

$$\begin{aligned} T_I : X(I) &\to D(I) \\ f &\to V_I U_I P_I f. \end{aligned}$$

Steps 1 to 3 show that, for $f \in X(I)$, we have

$$\frac{\|f\|_p}{C} \leqq \|T_I f\|_p \leqq C \cdot \|f\|_p,$$

where C is independent of f, $1 \leqq p \leqq \infty$, or I.

Moreover, the construction of T_I is such that $T_I f \in D(I)$, and T_I is surjective and linear.

PROPOSITION 6. *For $1 < p < \infty$ there exists $C_p > 0$ with the following property. Let $\mathscr{D}$ (with $|\mathscr{D}^*| < 1$) be a finite collection of dyadic intervals such that for $I \in \mathscr{D}$*

(i) *either $|G_1(I \mid \mathscr{D})| > |I|/2$ or $G_1(I \mid \mathscr{D}) = 0$,*

(ii) $|I| - |G_1^*(I \mid \mathscr{D})| > 0$.

Let $(\mathscr{F}_n)$ be the filtration induced by $\mathscr{D}$, then the Banach spaces $X^p_{\mathscr{D}}$ and $H^p[(\mathscr{F}_n)]$ are C_p-isomorphic.

PROOF. We put $\bar{\mathscr{D}} := \mathscr{D} \cup [0, 1]$. The isomorphism is defined on each interval of $\mathscr{D}$:

$$T: X^p_{\mathscr{D}} \to H^p[(\mathscr{F}_n)],$$

$$\left\{\sum_{I \in \mathscr{D}} \sum_{J \in G_1(I)} h_J a_J\right\} \to \left\{\sum_{I \in \mathscr{D}} T_I\left(\sum_{J \in G_1(I)} h_J a_J\right)\right\}.$$

By construction of T the following is easily observed.

(1) For $f \in X^2_{\mathscr{D}}$, $\|f\|_2/C \leqq \|Tf\|_2 \leqq \|f\|_2 \cdot C$.

(2) For $1 \leqq p \leqq \infty$, $n \in \mathbf{N}$ and $f \in X^p_{G_n}$,

$$\frac{\|f\|_p}{C} \leqq \|Tf\|_p \leqq \|f\|_p \cdot C.$$

(3) Let f be $\mathscr{F}_n$ measurable and let h be $\mathscr{F}_{n+1}$ measurable with $\mathbf{E}(h \mid \mathscr{F}_n) = 0$; then $\mathbf{E}(f \cdot h \mid \mathscr{F}_n) = 0$ and $T^{-1}(f \cdot h) = fT^{-1}(h)$.

It should be remarked here that T does not satisfy (3).

In [Ma] Section 4, an argument is given which shows that $T^{-1}: H^1[(\mathscr{F}_n)] \to X^1_{\mathscr{D}}$ is bounded. This argument is formulated there only for a very special collection of dyadic intervals. Nevertheless, a change of notation makes it work in our case too. Property (2) implies that $T^{-1}: H^2[(\mathscr{F}_n)] \to X^2_{\mathscr{D}}$ is bounded. Now we apply the interpolation theorem for H^p-spaces (cf. [C.W] Theorem D) to conclude that, for $1 < p \leqq 2$, $T^{-1}: H^p([\mathscr{F}_n)] \to X^p_{\mathscr{D}}$ is bounded.

Once we know that $T: X^p_{\mathscr{D}} \to H^p[(\mathscr{F}_n)]$, $1 < p \leqq 2$, is bounded, we are done by duality, because

$$(X^p_{\mathscr{D}})^* \quad \text{is} \quad X^q_{\mathscr{D}}$$

$$H^p[(\mathscr{F}_n)]^* \quad \text{is} \quad H^q([\mathscr{F}_n)]$$

where $1/p + 1/q = 1$.

An application of E. M. Stein's inequality permits us to show that $T: X^p_{\mathscr{D}} \to H^p[(\mathscr{F}_n)]$ is bounded.

The use of this inequality is possible because, for $I \in G_n$ and $f \in X(I)$, $|T_I f|$ can be dominated, pointwise, by the sum of conditional expectations of $|f|$. Indeed, the construction of T_I shows that

$$|T_I f| \leqq |f| + \mathbf{E}(|f| \mid \mathscr{C}_I) + 8\left(\int_I |f|\right)\cdot\frac{\chi_{F_1}}{|I|} + D_I(\mathbf{E}(|f| \mid \mathscr{C}_I)).$$

For the first three terms it is clear that they are dominated by conditional expectations. We take $\mathscr{E}^1_I = \mathscr{F}_n \cap I$, $\mathscr{E}^2_I := \mathscr{C}_I$ and $\mathscr{E}^3_I = \{\varnothing, I\}$.

Only the shift needs further considerations. We will now construct two different algebras of sets, both coarser than $\mathscr{C}_I$.

Let $F_1, \ldots, F_{m_0+1}$ be the atoms of $\mathscr{C}_I$. For $m_0 = 2n$ we put

$$\mathscr{E}^4_I := \mathscr{A}\{F_1 \cup F_2, F_3 \cup F_4, \ldots, F_{2n-1} \cup F_{2n}, F_{2n+1}\},$$

$$\mathscr{E}^5_I := \mathscr{A}\{F_1, F_2 \cup F_3, \ldots, F_{2n} \cup F_{2n+1}\}.$$

$\mathscr{E}^4_I$ and $\mathscr{E}^5_I$ will be the algebras which we are going to use.

We denote by M the multiplication operator induced by the characteristic function of the set $F_2 \cup F_4 \cup \cdots \cup F_{2n}$.

Using the abbreviation $h = \mathbf{E}(|f| \mid \mathscr{C}_I)$ we estimate as follows:

$$\begin{aligned} M(D_I h) &\leqq 2\mathbf{E}(M(D_I h) \mid \mathscr{E}^4_I) \\ &\leqq 4\mathbf{E}(h - M(D_I h) \mid \mathscr{E}^4_I) \\ &\leqq 4\mathbf{E}(h \mid \mathscr{E}^4_I) \\ &= 4\mathbf{E}(|f| \mid \mathscr{E}^4_I); \end{aligned}$$

$$\begin{aligned} D_I h - M(D_I h) &\leqq 2\mathbf{E}(D_I h - M(D_I h) \mid \mathscr{E}^5_I) \\ &\leqq 4\mathbf{E}(Mh \mid \mathscr{E}^5_I) \\ &\leqq 4\mathbf{E}(h \mid \mathscr{E}^5_I) \end{aligned}$$

$$= 4\mathbf{E}(|f| \mid \mathscr{E}_I^5).$$

Summing up, we obtain the following estimate:

$$|T_I f| \leqq 8 \sum_{j=1}^{5} \mathbf{E}(|f| \mid \mathscr{E}_I^j).$$

Just what we wanted!

If $m_0 = 2n + 1$, we have to put

$$\mathscr{E}_I^4 := \mathscr{A}\{F_1 \cup F_2, F_3 \cup F_4, \ldots, F_{2n+1} \cup F_{2n+2}\},$$

$$\mathscr{E}_I^5 := \mathscr{A}\{F_1, F_2 \cup F_3, \ldots, F_{2n} \cup F_{2n+1}, F_{2n+2}\}.$$

Finally, M will be the multiplication operator induced by the characteristic function of the set $F_2 \cup F_4 \cup \cdots \cup F_{2n+2}$, otherwise we do the same as above.

It remains to show how we will actually use the above estimate on T_I to bound T.

First, we introduce the global analogues of $\mathscr{E}_I^j$, $\mathscr{E}_n^j := \mathscr{A}\{\mathscr{E}_I^j : I \in G_n\}$.

Given $f := \Sigma_{I \in \mathscr{D}} a_I h_I$ we define

$$f_n := \sum_{I \in G_n} a_I h_I,$$

$$\Delta_n Tf := \mathbf{E}(Tf \mid \mathscr{F}_n) - \mathbf{E}(Tf \mid \mathscr{F}_{n+1}).$$

The definition of T gives the estimate

$$\sum_n |\Delta_n Tf|^2 \leqq C \sum_{j=1}^{5} \sum_{n-1}^{\infty} \mathbf{E}(|f_n| \mid \mathscr{E}_n^j)^2.$$

Hence we get from Stein's inequalities

$$\| Tf \|_{H^p[(\mathscr{F}_n)]}^p \leqq C \int \left(\sum_{n,j} (\mathbf{E}(|f_n| \mid \mathscr{E}_n^j))^2 \right)^{p/2}$$

$$\leqq C_p \int \left(\sum_{n,j} |f_n|^2 \right)^{p/2}$$

$$\leqq C_p \| f \|_{X^p_\bullet}^p . \qquad \square$$

Theorem 1 follows now from Proposition 6 and Lemma 3.

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